THE JOURNAL
OF THE
Indian Mathematical Society.

Vol. VIII.] DECEMBER 1916. [No. 6.

OFFICE OF THE REGIS[TRAR] REGIST[ERED] 20 DEC MA[DRAS]

PROGRESS REPORT.

The following gentlemen have been elected members :—

1. *Mr. Hiralal R. Kapadia, B.A.*—Asst. Teacher, Bhatia High School, 43, Navi Chawl, Bhagatwadi, Bhuleshwar, Bombay (at concessional rate);

2. *Mr. A. Gurumurti, B.A., L. T.*—Head Master, Board High School, Kavali (Nellore District.)

2. The attention of the members of the Society is drawn to the fact that all arrangements are being completed for holding the proposed meeting of the Society at Madras on 26th, 27th and 28th of December 1916. The detailed programme for this will be posted to our members shortly and the Committee most earnestly expects that the members will try their utmost to make this meeting a success.

3. *The Calcutta University Calendar, Part I* for 1916 has been received for the Library.

POONA,
30*th Nov.* 1916.

D. D. KAPADIA,
Hon. Joint Secretary.

161

The Homographic Function as an Operator.

By R. Vythynathaswamy.

[References: Forsyth:—*Theory of Functions of a Complex Variable*, Chs. XIX, XXI].

§ I.

The most general *algebraical* one-to-one correspondence between points on a line is given by the transformation

$$x' = S(x) = \frac{ax+b}{cx+d}.$$

The function S may be regarded as an operation which carries the point x to the point x'. If S_1, S_2 are to operations of this form, $S_2(S_1)$ is an operation of the same form; so that, the totality of these operations forms a 'group.' very operation S has a unique *inverse* operation of the same form, so that if $x' = S(x)$, we may write $x = S^{-1}(x')$. The *identical* operation of the group (corresponding to the transformation $S(x) = x$, may be denoted by the symbol 1.

If $S^2 = 1$, then S is an involution.

If S is not the identical transformation, the equation $S(x) = x$, determines two points which are *the fixed points* of S. Evidently these will also be the fixed points of any power of S or its inverse.

Theorem (1): *If a correspondence carries two distinct points into each other, it is an involution.*

Let $x = S(y)$, and $y = S(x)$.

Then $x = S(y) = S^2(x)$.

Since x, y are distinct, x is not a fixed point of S and therefore not a fixed point of S^2.

$\therefore$ $S^2 = 1$ identically;

i.e. S is an involution.

Theorem (2): *If I and SI be both involutions then IS is an involution and I contains the fixed points of S.*

For we have $(SI)^2 = 1$.

i.e. $SISI = 1$.

$\therefore$ $ISISI = I$.

$\therefore$ $(IS)^2 I = I$.

$\therefore$ $(IS)^2 = 1$.

Hence IS is an involution, (More generally if S_1, S_2 is an involution $S_2\,S_1$ is also an involution. See § II, below.)

Also if t_1, t_2 be the fixed points of S

$$(SI)^2\, t_1 = t_1.$$

$$\therefore \quad SI(t_1) = I^{-1}\,S^{-1}\,(t_1) = IS^{-1}(t_1) = I(t_1)$$

Hence $I(t_1)$ is a fixed point of S.

Now if $I(t_1)$ were equal to t_1 so that t_1, t_2 are the fixed points of I, then the fixed points of both SI and IS would be t_1, t_2. Hence, since these are involutions SI≡IS.

$$\therefore \quad S^2 = SI.IS = SI.SI = (SI)^2 = 1.$$

$\therefore$ S is an involution

Excluding this case, we have $I(t_1) = t_2$ or I contains the pair of fixed points of S.

Cor. *Conversely if I be an involution containing the fixed points of* S, S*I is an involution.*

For $$SI(t_1) = S(t_2) = t_2$$

and $$SI(t_2) = S(t_1) = t_1.$$

Hence by Theorem (1) SI is an involution.

Combining the theorem and the Corollary we see that the correspondence S^2 can be exhibited in an infinite number of waysas the product of two involutions SI and IS. We shall presently shew that any correspondence can be exhibited as a square so that this result is general.

Theorem (3) : *If two correspondences* S_1 S_2 *are permutable, either they have the same fixed points, or each of them is an involution containing the fixed points of the other.*

Let t_1, t_2 be the fixed points of S_2. Since S_1, S_2 are permutable by hypothesis, we have

$$S_2S_2(t_1) = S_2S_1(t_1)$$

$$\therefore \quad S_1(t_1) = S_2S_1(t_1),$$

so that $S_1(t_1)$ is a fixed points of S_2.

Hence, either $S_1(t_1) = t_1$ and $S_1?t_2) = t_2$

or $$S_1(t_1) = t_2 \text{ and } S_1(t_2) = t_1.$$

In the former case, S_1 and S_2 have the same fixed points. In the latter case, S_1 is an involution containing the fixed points of S_2. Since we might have assumed t_1, t_2 to be the fixed points of S_1 instead of S_2, it follows simultaneously that S_2 is an involution containing the fixed points of S_1. We further note that in this case S_1 S_2 $(=S_2S_1)$ is an involution and contains the fixed points of both S_1 and S_2, so that S_1, S_2 S_1S_2 *are three involutions each of which is determined by the fixed points of the other two.*

For $$S_1S_2\,(t_1) = S_1(t_1) = t_2,$$

and $$S_2S_1\,(t_2) = S_2(t_1) = t_1.$$

Hence by Theorem (1) S_1 S_2 is an involution evidently containing the fixed points of both S_1 and S_2.

Cor. (1): The converse of the theorem is true. That of the first part is proved under Theorem (5) below. We prove here the converse of the second part, *viz.* *Two involutions each containing the fixed points of the other are permutable and their product is the involution containing their fixed points.*

For if I_1, I_2 be the two involutions, the equation $I_1 I_2 = I_2 I_1$ is true for each of the fixed points of I_1 and I_2. Now, if $S(x) = S(x)$ for *three* distinct values of x, then $S = S_1$ identically.

Hence $$I_1 I_2 \equiv I_2 I_1$$

From this it is easy to see that three involutions whose product is unity would be represented in the plane by the sides of a triangle self-conjugate *w. r. t* the fundamental conic. (See Note on Involution and (1, 1) correspondence J.I.M.S., February 1916).

As an example consider the involutions

$$I_1 = (a_1\, a_2, a_3\, a_4), \; I_2 = (a_1\, a_3, a_2\, a_4), \; I_3 = (a_1\, a_4, a_3\, a_2)$$

It is at once verified that $I_1\, I_2\, I_3 = 1$.

Hence *the harmonic triangle of an inscribed quadrangle is self-conjugate.*

Cor. (2): *If the product of three involutions I_1, I_2, I_3 is an involution I, then all the four involutions have a common pair.*

For $I_1 . I_2\, I_3 = I =$ an involution.

$\therefore$ I_1 contains the fixed points of $I_2 I_3$ (Theorem (2))

Now the pair of fixed points of the product of two involutions is simply their common pair. Hence I_1, I_2, I_3 have a common pair. Evidently, I also contains this pair.

Theorem (4). *If* S *be a correspondence and p, q any two points, the involution $I \equiv [p, S(q)\,;\, q, S(p)]$ always contains the fixed points of* S.

For $$IS(p) = q \text{ and } IS(q) = p.$$

Hence by Theorem (1) IS is an involution

$\therefore$ I contains the fixed points of S (Th. 2).

Cor. Let $S \equiv \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ *i.e.* Let S carry p, q, r respectively into $p'\, q'\, r'$. From the present theorem we see that the involution $I_1 = (pq', p'q)$, $I_2 = (qr', q'r)$ $F_3 = (rp', r'p)$ have a common pair, *viz.* the pair of fixed points of S.

This is evident otherwise. For, $I_1 I_4 I_3(p) = p'$ and $I_1\; I_2\; I_3\; (p') = p$. Hence $I_1 I_2 I_3$ is an involution.

$\therefore$ $I_1 I_2 I_3$ have a common pair [Th. (3). Cor. 2]. Transforming this result to the plane we have Brianchon's theorem.

Theorem (5). *If two correspondences have the same fixed points they are permutable.*

Let S_1, S_2 be the two correspondences.

Let $x_1 = S_1(x)$, $x_2 = S_2(x_1)$, $x_3 = S_2(x)$, $x' = S_1(x_3)$;

so that S_1 is $\begin{pmatrix} x & x_3 \\ x_1 & x' \end{pmatrix}$ and S_2 is $\begin{pmatrix} x_1 & x \\ x_2 & x_3 \end{pmatrix}$

By Theorem (4) the involutions $(x_1x_3,\ xx')$, $(x_1x_3,\ xx_2)$ contain the fixed points of S_1 and S_2 respectively. If these fixed points are the same, the involutions become identical since they have another common pair x_1x_3.

Hence $\qquad x' = x_2$. *i.e.* $S_1S_2(x) = S_2S_1(x)$.

Thus S_1, S_2 are permutable.

§ 11.

The parameter of a correspondence. The fixed points may be supposed to define the *position* of the correspondence in the line. We now proceed to find a function which is an appropriate quantitative measure of a correspondence S; *i.e.*, which satisfies the relation

$$f(S^n) = \{f(S)\}^n.$$

By a well-known theorem in the Theory of Linear Transformation the determinant of the matrix of S^n is the nth power of the determinant of the matrix of S. But the difficulty in taking $\triangle$ the determinant of S $\left(=\dfrac{ax+b}{cx+d}\right)$ for our function, is that $\triangle$ depends on the absolute values of $a\ b\ c\ d$, while S depends only on their ratios.

Let S be reduced to the form $\dfrac{kx+t^2}{x+k}$, so that the fixed points are $\pm t$. Let S′ be another correspondence having the same fixed points, so that

$$S'(x) = \frac{k'x+t^2}{x+k'}.$$

Let

$$S'' = S'S = \frac{k''x+t^2}{x+k''}.$$

$$\text{Then } k'' = S''(\infty) = S'S(\infty) = S'(k) = \frac{kk'+t^2}{k+k'}.$$

$$\therefore \qquad \frac{k''}{t} = \frac{kk'+t^2}{t(k+k')}; \text{ so that } \frac{k''+t}{k''-t} = \frac{k'+t}{k'-t}\cdot\frac{k+t}{k-t}.$$

Hence if $\dfrac{k+t}{k-t}$ be termed the parameter of S (called 'multiplier' by Forsyth) we have the theorem:—

The parameter of the product of two correspondences having the same fixed points is the product of their parameters.

Hence: *if* λ *be the parameter of* S, λ^n *is the parameter of* S^n.

The parameter λ is verified to be the cross-ratio of the fixed points of S and any pair of correspondents of S.

It can also be shewn that the parameter of S $\left(=\frac{ax+b}{cx+d}\right)$ is the ratio of the latent roots of the matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, *i.e.*, of the roots of the equation

$$\begin{vmatrix} a-\rho & b \\ c & d-\rho \end{vmatrix}=0.$$

The value of the parameter in terms of $a\ b\ c\ d$ is given by

$$\lambda=\frac{(a+d)^2-2\triangle+(a+a)\sqrt{(a+d)^2-4\triangle}}{2\triangle}$$

where $\triangle=ad-bc.$

It must be noted that the definition of the parameter involves a definite order of the fixed points. If this order is changed the parameter will be changed into its reciprocal.

(1) A correspondence is uniquely determined if the fixed points and the parameter be known.

(2) Since λ^n is the parameter of S^n when n is integral, *we may define* S^n, *when* n *is not integral, to be the determinate correspondence which has the same fixed points as* S *and whose parameter is* λ^n.

The present theorem then proves that $S^m.\ S^n=S^{m+n}$ for all values of m and n.

If S is given in the form $\frac{kx+t^2}{x+k}$, the present theorem gives at once

$$S^n(x)=\frac{(\lambda^n+1).\,xt+t^2(\lambda^n-1)}{(\lambda^n-1)x+(\lambda^n+1)\,t}, \qquad \left(\lambda=\frac{k+t}{k-t}\right)$$

for all values of n.

(3) *Periodic correspondences.*

A correspondence is said to have a period n, when its nth power is the identical correspondence. The parameter of such a correspondence would be an nth root of unity; (k $(=S(\infty))$ being infinite for the identical correspondence, its parameter is 1). The number of correspondences of period n and no smaller period, with given fixed points, is equal to the number of integers less than n and prime to it. The forms of these are easily found as well as the condition that a correspondence may have the period n.

(4) *Uniparametric correspondences.*

This is the class of correspondences which are not periodic but are *nearly periodic.* The modulus of the parameter of such a correspondence S would be unity and the amplitude would be incommensurable with π. What is meant by saying that S is nearly periodic, is that an integer n can always be chosen so that S^n is as nearly equal to the identical correspondence as we please. As a matter of fact the set of points in a line defining segments commensurable with a given segment has every point as a limit-point.

We note that every integral power of a uniparametric correspondence is uniparametric and that a correspondence the modulus of whose parameter is unity, is uniparametric unless it is periodic.

(5) *The correspondence $S_1^{-1} SS_1$ is termed the transform of S by S_1; and the paramcter of any transform of S is equal to the parameter of S.*

Let $T = S_1^{-1} SS_1$ and let x, y be the fixed points of S. The fixed points of T are $S_1^{-1}(x)$, $S_1^{-1}(y)$.

We have,

$$\begin{aligned} \text{parameter of } T &= \text{cross-ratio} \{ S_1^{-1}(x), S_1^{-1}(y); p, S_1^{-1}SS_1(p) \} \\ &= \text{cross-ratio} \{ x, y, S_1(p), SS_1(p) \} \\ &= \text{cross-ratio} \{ x, y, q, S(q) \} \\ &= \text{parameter of } S. \end{aligned}$$

As a corollary it follows that the parameter of S_1S_2 (S_1 and S_2 not having the seme fixed points) is equal to the parameter of S_2S_1, though neither of these would be equal to the product of the parameters of S_1 and S_2. [Theorem (2) is an example of this property].

For $S_2S_1 = S_1^{-1}(S_1S_2)S_1 =$ a transform of S_1S_2.

Hence follows the interesting property : *that the parameter of the product of a number of correspondences is the same so long as the cyclical order of the product is the same.*

§III.

Sub-groups of the general Homographic Group :

(1) *A continuous sub-group of the general homographic group must either consist of all correspondences having the same fixed points, or of all correspondences having a single point as one of their fixed points.*

Let S_1 belong to the continuous sub-group K. Then $S_1 \pm m$, where m is an integer belongs to K.. If K contains no other correspondence with the same fixed points as S_1, then K cannot be a *continuous* sub-group. For, we cannot find an *integer n* so that

$$S_1^n(x) = x + dx$$

i.e., $S_1^n = 1 + \delta$ (where S_1 is the parameter of S_1 and dx and δ are arbitrarily small).

Hence K does not contain the infinitesimal transformation.

It is easy to see that this necessarily leads to the conclusion that K must contain *all correspondences* having the same fixed points as S_1.

[It might appear that this reasoning would fail if S_1 were uniparametric. This exception would not however arise, if we insist that *a continuous group must contain not merely some but all possible infinitesimal transformations* i.e., must contain a transformation which carries a point x into any *assigned* near point $x+dx$. On this understanding it is easy to shew that the group $S_1 \pm m$ (S_1 being uniparametric) is discontinuous. For, though we can choose n so that S_1^n is an infinitesimal transformation, yet there are an infinite number of infinitesimal transformations with the same fixed points, which are not expressible in the form S_1^n; for example, the infinitesimal *periodic* transformations are not so expressible. As a matter of fact the set of points on the unit circle in the Argand diagram, representing the parameters of *all* uniparametric correspondences, is a point-wise discontinuous set; the points of discontinuity (the points whose amplitudes are rational multiples of π) forming evidently an enumerable aggregate.]

Let now, K contain, if possible, a correspondence S_2 not having the same fixed points as S_1. We have to consider two cases:—

Case I.—S_2 has both its fixed points different from those of S_1.

Repeating the argument above we shew that every correspondence P_2 having the same fixed points as S_2, belongs to K. Hence if P_1 has the same fixed points as S_1, every correspondence of the form $(P_1P_2)^t$ belongs to K.

It can be easily shewn that, by properly choosing the parameters P_1 and P_2, we can make $(P_1P_2)^t$ have any two given points as its fixed points; and by properly choosing t we can make it have any parameter we choose; thus K coincides with the general homographic group and is not a sub-group.

The supposition of *Case I* is therefore impossible.

Case II.—S_2 has one fixed point, L, common with S_1. We shew, on the same principle as before that K, if it is to be continuous, must contain all correspondences one of whose fixed points is L.

Hence the theorem is proved.

We note that in the first case (when Kcontains no correspondence (S_2), K is one—dimensional *i.e.* the form of its most general infinitesimal transformation contains one arbitrary constant. In the latter case K is two-dimensional.

(2) The group $S^{-m}.S^{-m+1}...1\ S\ S^2$ is a discontinuous subgroup of finite or infinite order according as S is or is not periodic.

There is a remarkable type of *a cyclic* subgroup of finite order 4. This is the group consisting of the identical correspondence and three involutions each of which contains the fixed points of the other two (*Cf.* Th. (3)). The existence of this type of groups is closely related to (and may in fact be used to prove) the possibility of algebraic solution of the general biquadratic. It is fairly certain that this is the only possible type of acyclic subgroups of finite order. I have not howev r seen it proved anywhere.

§ IV.

The Theory of Distance.

(1) The well-known projective definition of distance may be stated concisely in terms of a fundamental correspondence S.

Let the fixed points of S be $t_1\ t$; (S is to be neither periodic nor uniparametric.)

Let
$$\begin{pmatrix} t_1 & t_2 & p \\ t_1 & t_2 & q \end{pmatrix} = S.^{\lambda}$$

(Note that any two correspondences with the same fixed points can be expressed as powers of one another).

The distance pq may be defined to be the index λ. This definition is in conformity with our ordinary ideas.

For if
$$\begin{vmatrix} t_1 & t_2 & p \\ t_1 & t_9 & q \end{vmatrix} = S^{\lambda}, \text{ then } \begin{vmatrix} t_1 & t_2 & q \\ t_7 & t_2 & p \end{vmatrix} = S^{-\lambda}$$
so that $pq+qp=\lambda-\lambda=0$.

Also if
$$\begin{vmatrix} t_1 & t_2 & p \\ t_1 & t_2 & q \end{vmatrix} = S^{\lambda}, \begin{vmatrix} t_1 & t_2 & q \\ t_1 & t_2 & r \end{vmatrix} = S^{\mu},$$
then
$$\begin{vmatrix} t_1 & t_2 & p \\ t_1 & t_2 & r \end{vmatrix} = S^{\lambda}.S^{\mu} = S^{\lambda}, {}^{+\mu}$$
so that $pq+qr=pr$.

Further the parameter of S^n is the nth power of the parameter of S and therefore tends either to zero or infinity as $n \to \infty$.

Hence $S^{\infty}(x)=t_1$ or t_2, where x is any point.

Thus t_1, t_2 are at an infinite distance from all other points.

The distance as thus defined will be multiple-valued, for we can always find x from the equation
$$S^x=1, \textit{ i.e.}, s^x=1$$

where s is the parameter of S. x as thus determined will not in general be real. But if S is negative

$$x=\frac{2ik\pi}{\log s}=\frac{2k}{\log(-s)}=\text{a real number}.$$

For example, if $s=-e^{a/\pi}$, the distance is multiple-valued to the extent of a multiple of $2\pi/a$. This corresponds to distance on the sphere.

(2) *This definition of distance is the most general possible one in conformity with our spatial ideas.* (Klein.)

Distance is perceived as something connected with two points, which is preserved by a class (C) of homographic transformations which are the congruent transformations. The class (C) is a continuous one-dimensional subgroup of the homographic group. For, (C) is a group since if C_1 and C_2 severally preserve distance, $C_1 C_2$ preserves distance and (C) is continuous and one-dimensional since a unique C belonging to (C) can be found to carry any arbitrary point into any other. Hence by § III (1), (C) consists of all correspondences having the same fixed points.

If the C's be correspondences belonging to C and (pq) represent the distance pq, we have

$$(pC(p))=\{C_1(p)C_1C(p)\}, \text{ since } C_1 \text{ preserves distance}$$
$$=\{C_1(p)CC_1(p)\}, \text{ since } C_1 \text{ and } C \text{ have the same fixed points and are therefore permutable}$$
$$=\{qC(q)\}, \text{ whatever } q \text{ may be.} \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Hence, assuming the fundamental property of distance, *viz.* $(pq)+(qr)=(pr)$, we have

$$\{pC^n(p)\}=\{pC(p)\}+\{C(p)C^2(p)\}+\ldots\ldots+\{C^{n-1}(p)C^n(p)\}$$
$$=n\{pC(p)\} \text{ by (i)}$$

Putting $C^n=C_1$ in this result, we easily prove

$$\{pC^n(p)\}=n\{pC(p)\} \text{ for all values of } n.$$

If $\{pC(p)\}$ is the unit of distance, we see that $\{pC^n(p)\}=n$; hence, *if pq is the unit of distance and C is the unique correspondence of (C) which carries p to q, any distance LM must be defined as the number n, where $M=C^n(L)$.*

For a proof on different lines see Whitehead's *Universal Algebra*, Vol. (I), page 353.

Several of the theorems in this paper have analogues in the general Linear Transformation. These I shall discuss in a future paper.

SHORT NOTES.

Note on sn, cn, dn of $(u_1 + u_2 + u_3)$.

Symmetrical results for the expression of sn, cn, dn of $(u_1+u_2+u_3)$ in terms of those of u_1, u_2, u_3 given by Dr. Glaisher and Prof. Cayley are quoted in Whittaker's *Modern Analysis*, 2nd Edn, Chap. XXII, Misc. Ex., 19 and 20.

The following additional symmetrical forms may be of interest.

(1)

$$\text{sn}\,(u_1+u_2+u_3)=\frac{X}{W},$$

$$\text{cn}\,(u_1+u_2+u_3)=\frac{Y}{W},$$

$$\text{dn}\,(u_1+u_2+u_3)=\frac{X}{W};$$

where

$$X=\sum\left[\frac{\partial s_3}{\partial u_3}.\ s_1s_2(s_1^2-s_2^2)\right]$$

$$Y=-\sum\left[\frac{\partial c_3}{\partial u}.\ c_1c_2(c_1^2-c_2^2)\right]$$

$$Z=-\frac{1}{k^4}\sum\left[\frac{\partial d_3}{\partial u_3}.\ d_1d_2(d_1^2-d_2^2)\right]$$

$$W=-\sum\left[s_1.\frac{\partial s_1}{\partial u_1}(s_2^2-s_3^2)\right]$$

$$=\sum\left[s_1.\frac{\partial s_1}{\partial u_1}(c_2^2-c_3^2)\right]$$

$$=\frac{1}{k^2}\sum\left[s_1\frac{\partial s_1}{\partial u_1}(d_2^2-d_3^2)\right]$$

$$=\sum\left[s_1s_2\left(s_1.\frac{\partial s_2}{\partial u_2}-s_2.\frac{\partial s_1}{\partial u_1}\right)\right]$$

$$=\sum\left[c_1c_2\left(c_1.\frac{\partial c_2}{\partial u_2}-c_2\frac{\partial c_1}{\partial u_1}\right)\right]$$

$$=\frac{1}{k^4}\sum\left[d_1d_2\left(d_1\frac{\partial d_2}{\partial u_2}-d_2.\frac{\partial d_1}{\partial u_1}\right)\right],$$

the summation extending to the suffixes 1, 2, 3.

Proof. If $u_1+u_2+u_3+u_4=0$ we can deduce from Jacobi's Fundamental Theta-function formulae that

$$[1133]-[3311]+[4422]-[2244]=0$$

where $[1133]$ means $I_1(u_1).\ I_1(u_2).\ I_3(u_3).\ I_3(u_4)$; etc. the notation I employed for the Theta function corresponding to that of Tannery and Molk followed by Whittaker and Watson in their *Modern Analysis*.

Hence $s_1\, s_2\, d_3.d_4 - d_1\, d_2\, s_3.s_4 + c_3.c_4 - c_1\, c_2 = 0,$

or $d_1\, d_2\, s_3.s_4 - c_3.c_4 - s_1\, s_2.d_3.d_4 + c_2\, c_2 = 0.$ ... (i)

Similarly $d_2\, d_3\, s_1.s_4 - c_1.c_4 - s_2\, s_3\, d_1.d_4 + c_2\, c_3 = 0,$ ... (ii)

$d_3\, d_1\, s_2.s_4 - c_2.c_4 - s_3\, s_1\, d_2.d_4 + c_3\, c_1 = 0.$ ... (iii)

Hence, since $\operatorname{sn}(u_1+u_2+u_3) = -\operatorname{sn} u_4 = -s_4$

$\operatorname{cn}(u_1+u_2+u_3) = \operatorname{cn} u_4 = c_4.$

$\operatorname{dn}(u_1+u_2+u_3) = \operatorname{dn} u_4 = d_4.$

We have from (i), (ii), (iii),

$$\frac{\operatorname{sn}(u_1+u_2+u_3)}{\begin{vmatrix} c_3, & s_1 s_2 d_3, & c_1 c_2 \\ c_1, & s_2 s_3 d_1, & c_2 c_3 \\ c_2, & s_3 s_1 d_2, & c_3 c_1 \end{vmatrix}} = \frac{-\operatorname{cn}(u_1+u_2+u_3)}{\begin{vmatrix} d_1 d_2 s_3, & s_1 s_2, d_3, & c_1 c_2 \\ d_2 d_3 s_1, & s_2 s_3 d_1, & c_2 c_3 \\ d_3 d_1 s_2, & s_3 s_1 d_2, & c_3 c_1 \end{vmatrix}}$$

$$= \frac{\operatorname{dn}(u_1+u_2+u_3)}{\begin{vmatrix} d_1 d_2 s_3, & c_3, & c_1 c_2 \\ d_2 d_3 s_1, & c_1, & c_2 c_3 \\ d_3 d_1 s_2, & c_2, & c_3 c_1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} d_1 d_2 s_3, & c_3, & s_1 s_2 d_3 \\ d_2 d_3 s_1, & c_1, & s_2 s_3 d_1 \\ d_3 d_1 s_2, & c_2, & s_3 s_1 d_2 \end{vmatrix}}$$

Hence we obtain the results written down above.

(2)
$$\operatorname{sn}(u_1+u_2+u_3) = \frac{L}{R},$$

$$\operatorname{cn}(u_1+u_2+u_3) = \frac{M}{R},$$

$$\operatorname{dn}(u_1+u_2+u_3) = \frac{N}{R};$$

where

$$L = \Sigma s_1^3 c_2 d_2 c_3 d_3 - s_1 s_2 s_3 [c_2^2 d_1^2 + c_1^2\, d_3^2 + c_3^2 d_2^2]$$
$$= \Sigma s_1 c_2 d_2 c_3 d_3 - c_1 c_2 c_3 \Sigma d_1 d_2 s_3 c_3$$
$$- s_1 s_2 s_3 [3 - (1+k)^2 \Sigma s_1^2 + k^2 \Sigma s_1^2 s_3^2]$$
$$M = c_1 c_2 c_3 [s_1^2 d_2^2 + s_2^2 d_3^2 + s_3^2 d_1^2] - \Sigma c_1^3 s_2 d_2 s_3 d_3$$
$$= c_1 c_2 c_3 [\Sigma s_1^2 - k^2 \Sigma s_1^2 s_2^2] - \Sigma c_1 s_2 d_2 s_3 d_3 + s_1 s_2 s_3 \Sigma d_1 d_2 s_3 c_3\,;$$

$$N = d_1 d_2 d_3 [c_1^2 s_1^2 + c_2^2 s_3^2 + c_3^2 s_1^2] - \Sigma d_1^3 c_2 s_2 c_3 s_3.$$
$$= d_1 d_2 d_3 [\Sigma s_1^2 - \Sigma s_1^2 s_2^2] - \Sigma d_1 c_2 s_2 c_3 s_3 + k^2 s_1 s_2 s_3 \Sigma s_1 d_1 c_2 c_3$$
$$R = s_1^2 c_2^2 d_3^2 + s_2^2 c_3^2 d_1^2 + s_3^2 c_1^2 d_2^2 - \Sigma s_1 c_1 d_1 s_2 c_2 d_2$$
$$= \Sigma s_1^2 - (1+k^2) \Sigma s_1^2 s_2^2 + 3k^2 s_1^2 s_2^2 s_3^2 - \Sigma s_1 c_1 d_1 s_2 c_2 d_2;$$

the summation referring to the suffixes 1, 2, 3.

Proof. Suppose $u_1 + u_2 + u_3 + u_4 = 0$,

Then either from Jacobi's Fundamental Theta-function formulae, or from the relations

$$\frac{s_1 c_4 + s_4 c_1}{d_1 + d_4} + \frac{s_3 c_2 + s_2 c_3}{d_3 + d_2} = 0$$

and

$$\frac{s_1 c_4 - s_4 c_1}{d_1 - d_4} + \frac{s_3 c_2 - s_2 c_3}{d_3 - d_2} = 0.$$

we obtain $\quad c_1 d_2 . s_4 + s_1 d_3 . c_4 + s_2 c_3 . d_4 + s_3 c_2 d_1 = 0;$

similarly $\quad c_2 d_3 . s_4 + s_2 d_1 . c_4 + s_3 c_1 . d_4 + s_1 c_3 d_2 = 0,$

$$c_3 d_1 . s_4 + s_3 d_2 . c_4 + s_1 c_2 . d_4 + s_2 c_1 d_3 = 0.$$

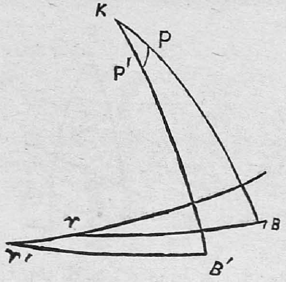
Solving the last three equations for s_4, c_4, d_4 we obtain the results stated.

F. H. V. Gulasekaram.

Precession and Nutation.

The following method of finding the charges in the right ascension and declination of a star on account of precession and nutation may be of interest to those preparing for an examination.

Let $P\Upsilon B$ be a quadrantal triangle on the celestial sphere, so that B has a right ascension, 90° P being the pole of the equator. Let P', Υ', B' denote their positions as altered by precession and nutation.



If α, δ be the co-ordinates of a star S then the direction cosines of OS are $\cos S\Upsilon$, $\cos SB$, $\cos SP$; *i.e.*, $\cos\alpha \cos\delta$, $\sin\alpha\cos\delta$, $\sin\delta$ respectively.

We know that θ, the angle between two vectors (l, m, n) and (l', m', n') is $\cos^{-1}(ll'+mm'+nn')$.

Also $\Upsilon' KP' = \frac{\pi}{2} = \Upsilon KP$. $\therefore$ $\Upsilon'\Upsilon = P'KP$; $KP = \omega$; $KP' = \omega'$.

(i) Now $\sin\delta' = \cos SP' = \sin\delta . \cos PP' + \sin\alpha \cos\delta . \cos BP'$
$+ \cos\alpha \cos\delta . \cos\delta . \cos \Upsilon P'$
$= \sin\delta (\cos\omega . \cos\omega' + \sin\omega . \sin\omega' \cos k)$
$+ \sin\alpha \cos\delta(-\cos\omega' \sin\omega + \sin\omega' \cos\omega \cos k)$
$+ \cos\alpha \cos\delta(\sin\omega' . \sin k)$

(ii) $\sin\alpha, \cos\delta' = \cos SB' = \sin\delta \cos PB' + \sin\alpha \cos\delta \cos BB'$
$+ \cos\alpha \cos\delta \cos \Upsilon B'$
$= \sin\delta(-\cos\omega \sin\omega' + \sin\omega . \cos\omega' \cos k)$
$+ \sin\alpha \cos\delta(\sin\omega . \sin\omega' + \cos\omega \cos\omega' \cos k)$
$+ \cos\delta \cos\delta(\cos\omega' . \sin k)$

(iii) $\cos\alpha' \cos\delta' = \cos S\Upsilon' = \sin\delta . \cos P\Upsilon' + \sin\alpha \cos\delta$
$B\Upsilon' + \cos\alpha \cos\phi \cos P\Upsilon B$
$= \sin\delta(-\sin\omega . \sin k)$
$+ \sin\alpha \cos\delta(\sin k \cos\omega) + \cos\alpha \cos\delta \cos . k.$

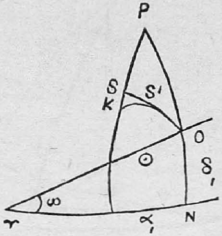
The rest of the working is the same as that given in Ball's. *Spherical Astronomy*, p. 189.

K. K. RANGANATHA AIYAR.

Annual Parallax.

The following simple geometrical method may be substituted for Art iii, p. of 330 Ball's *Spherical Astronomy* to find the effect of parallar on the right ascension and declination of a fixed star.

The effect of parallax is to throw the star from its mean place towards the sun through an angle $=\sigma$ sin E where σ is the amount of annual parallax and E is the angular distance between the star and the sun.



Let P be the pole of the equator, S, S′ the true and apparent positions of the star.

Then $SS' = \sigma \sin SO$, where O is the sun.

If (α, δ) be the R.A., and declination of the star, then

(1) $\alpha' - \alpha = S'K \sec. \delta$

$= \sigma \sin SO, \sin S'SK \sec \delta$

$= \sigma \sec \delta \sin (\alpha_1 - \alpha) \cos \delta_1$

[α_1, δ_1 being the coords, of O].

$= \sigma \sec \delta \{ \sin \alpha_1 \cos \delta_1 \cos \alpha - \cos \alpha_1 \cos \delta_1 \sin \alpha \}$

$= \sigma \sec \delta \{ \cos \alpha \sin \odot \cos \omega - \cos \odot \sin \alpha \}$,

since $\sin \alpha_1 \cos \delta_1 = \sin \alpha_1 \cos \odot \sec \alpha_1 = \cos \omega \sin \odot$,

and $\cos \alpha_1 \cos \delta_1 = \cos \odot$.

(2) $\delta' - \delta = -SK = SS' \cos S'SP$

$= \sigma \sin SO \cos OSP$

$= \sigma (\cos OP - \cos OS \cos SP) \operatorname{cosec} SP$

$= \sigma (\sin \delta_1 - \cos OS. \sin \delta) \sec \delta$

$= \sigma [\sin \delta_1 \cos \delta - \cos \delta_1 \sin \delta_1 \cos (\alpha_1 - \alpha)]$

from the $\triangle$SPO.

$= \sigma \{ \sin \odot \sin \omega \cos \delta - \sin \delta \cos \alpha \cos \odot$

$- \sin \delta \sin \alpha \cos \omega \sin \odot \}$

since $\sin \delta_1 = \sin \odot \sin \omega$

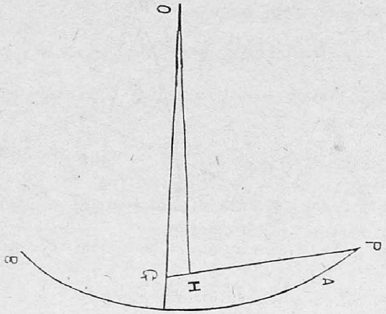
$\cos \delta_1 \cos \alpha_1 = \cos \odot$

$\cos \delta_1 \sin \alpha_1 = \sin \odot \cos \omega$; from (1).

C. Krishnamachary.

Centroid of a Uniform Circular Arc.

Let n equal particles of unit mass be placed at equal intervals α along the arc AB whose centre is O, so that $A\hat{O}B=(n-1)\alpha$; and let x = distance of the centroid G from O. Add an equal particle at P where $AP=\alpha$. Then the centroid of the $(n+1)$ particles is in PG and must be H such that $H\hat{O}G=\frac{1}{2}\alpha$, since, by symmetry, OG, OH are the bisectors of the angles AOB, POB respectively.



$$\therefore \qquad \tan\tfrac{1}{2}\alpha=\frac{a}{a\cos\left[\frac{1}{2}(n-1)+1\right]\alpha+nx},$$

by the usual formula for 'centre of mass'.

Hence $nx \sin\frac{1}{2}\alpha=a\sin\left[(\frac{1}{2}n+\frac{1}{2})\alpha-\frac{1}{2}\alpha\right]$

$$=a\sin\tfrac{1}{2}n\alpha.$$

$$\therefore\ n=a\sin(\tfrac{1}{2}n\alpha)/(n\sin\tfrac{1}{2}\alpha).$$

Cor. When n is infinite, since $(n-1)\alpha=A\hat{O}B=2\theta$, $\frac{1}{2}n\alpha=\theta$; that is, the centroid of the uniform circular arc AB is given by

$$x=a(\sin\theta)/\theta.$$

M. T. NARANIENGAR.

SOLUTIONS.

Question 649.

(S. Malhari Rao, B.A.) :—A gentleman has one-third of an acre of vacant land round his house. He wishes to divide it into five different plots to grow five different flowers. What must be the areas of these plots so that they may be in A. P., and each of them may be contained an exact number of times in an acre? Shew that there cannot be more than one set of values for the areas.

Remarks by Eric H. Neville.

The solution is much abbreviated if we take into account from the start the condition

$$\frac{1}{y}-\frac{1}{x}<\frac{1}{15}$$

which in the solution on page 60 of the *Journal* is used only at the end. This condition, in virtue of the equation

$$\frac{1}{y}+\frac{1}{x}=\frac{2}{15},$$

is equivalent to $\qquad x<3y,$

and we have also $\qquad y<x.$

(i) If a is an integer and

$$x=15a=(2a-1)y,$$

then $2a-1$ is an odd integer between 1 and 3, which is impossible.

(ii) If a, b are integers, and

$$x=3a,\ y=5b,\ 3a+5b=2ab,$$

then since

$$5b/a=2b-3,$$

$5b/a$ is an odd integer which being equal to $3y/x$ is less than 3 and greater than 1, which again is impossible.

(iii) If a is an integer and

$$y=15a=(2a-1)\,x,$$

then $2a-1$ is an integer less than 1.

(iv) If a, b are integers and

$$x=5a,\ y=3b,\ 5a+3b=2ab,$$

then since

$$3b/a=2b-5,$$

$3b/a$ is an odd integer, which being equal to $5y/x$ is less than 5 and greater than 5/3, and therefore is necessarily 3. Thus the *sole* solution fulfilling the conditions assigned, is given by

$$a=b=4,\ x=20,\ y=12.$$

Question 691.

(A. Narsinga Rao) :—A marble slab of n pound breaks into k pieces with which a tradesman finds himself able to weigh goods from 1 to n pounds (fraction excluded). Shew that the least value of λ is the smallest integer satisfying the relation $3^k \geq 2n+1$.

What are the weights of the several pieces ?

Additional solution by T. P. Trivedi, M.A., LL.B.

It is obvious that when weighing, each piece can be dealt with in 3 ways. It may be put in the first scale pan, or in the second, or it may not be used at all ; thus the k pieces can be dealt with in 3^k ways ; excluding the case when no piece is used, the number of weighings is 3^k-1 ; but since the weights in the two pans can be interchanged, the number of *distinct* weighings possible is $\frac{3^k-1}{2}$ and this must be not$<n$.

$$\therefore \qquad \frac{3^k-1}{2} \nless n, \text{ or } 3^k \nless 2n+1.$$

Again, since there are 3 ways of dealing with the weights, it is obvious that the weights will form a G. P. of which 3 is the common ratio. Hence the weights will be 1, 3, 9, 27 etc.

Taking 4 weights (1, 3, 9, 27), the number of *distinct* weighings is $\frac{3^4-1}{2}+40$; and since the maximum that can be weighed is $1+3+9+27 =40$, and the minimum is 1, all the weighings from 1 to 40 pounds are possible in one and one way only.

Question 739.

(S. Ramanujan) :—Show that

$$\int_0^\infty e^{-nx}(\cot x+\coth x)\sin nx\, dx=\frac{\pi}{2}\left(\frac{1+e^{-n\pi}}{1-e^{-n\pi}}\right)^{(-1)^n}$$

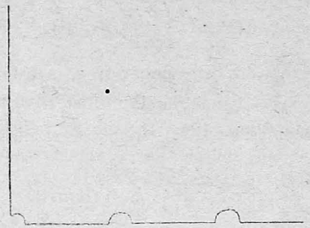
for all positive integral values of n.

Solution by A. C. L. Wilkinson.

Consider the integral $\int e^{-n(1-i)z}\cot z\, dz$ over the following contour :—

The real axis of x indented with semi-circular contours of radii $\rho \to 0$ surrounding the points π, 2π,... ; an infinitely great quarter

circle, the imaginary axis, and a quarter circle of radius ρ surrounding the origin. By Cauchy's theorem this is zero.



For the infinitely great quarter circle the integral becomes comparable with

$$\int_0^{\frac{1}{2}\pi} e^{-n\mathrm{R}(\cos\Theta+\sin\Theta)}\,\mathrm{R}\,d\Theta,$$

and this part is zero.

We thus get

$$\int_\rho^{\pi-\rho}+\int_{\pi+\rho}^{2\pi-\rho}+\int_{2\pi+\rho}^{3\pi-\rho}+\ldots e^{-nx}(\cos n\,x+i\sin nx)\cot x\,dx$$

$$-\int_0^\infty e^{-nx}(\cos nx-i\sin\ nx)\coth x\,dx+\underset{\rho\to 0}{\mathrm{Lt}}\int_{\frac{1}{2}\pi}^{0} i\,d\Theta$$

$$+\sum_{r=1}^{r=\infty}\underset{\rho\to 0}{\mathrm{Lt}}\int_\pi^0 e^{-n(1-i))\,r\pi}\,i\,d\Theta=0.$$

Equate the imaginary part and since $\int e^{-nx}\sin n\,x\cot x\,dx$ is no longer infinite at $x=r\pi$, we can write

$$\int_0^\infty e^{-nx}(\cot x+\coth x)\sin n\,x\,dx$$

$$=\frac{\pi}{2}\left(1+2\Sigma e^{-nr\pi}\cos nr\pi\right)$$

$$=\frac{\pi}{2}\left(\frac{1+e^{-n\pi}}{1-e^{-n\pi}}\right)^{(-1)^n}.$$

Question 740.

(S. RAMANUJAN) :—If

$$\phi(x)=\frac{e^x\{[x]^2\}!}{[x]}-2\pi x$$

where $[x]$ denotes the greatest integer in x, show that $\phi(x)$ is a continuous function of x for all positive values of x, and oscillates from $\frac{\pi}{3}$ to $-\frac{\pi}{6}$ when x becomes infinite. Also differentiate $\phi(x)$.

Solution by A. C. L. Wilkinson.

The function is continuous at $x=r$, for

$$\underset{h\to o}{\text{Lt.}}\left\{\frac{er-1)(!^{-h})}{(r-h)^{r-1}}\right\}^2-2\pi(r-h);\ \left\{\frac{e^r.r!}{r^r}\right\}^2-2\pi r;$$

and $$\underset{h\to o}{\text{Lt.}}\left\{\frac{e^{r+h}.r!}{(r+h)^r}\right\}^2-2\pi(r+h)\ \text{are the same}$$

The progressive and regressive differential coefficients at these points are however different.

Consider $x=n$ when n is large

$$\phi(n)=\left\{\frac{e^n(n-1)!}{n^{n-1}}\right\}^2-2\pi n=2\pi n\left\{1+\frac{1}{6n}+\cdots\right\}-2\pi n$$

$$=\frac{\pi}{3},\ \text{by use of Sterling's theorem};$$

and so for $x=n\pm\epsilon$, since (x) is continuous, $\phi(n\pm\epsilon)=\frac{\pi}{3}$.

$$\text{Consider}\ \frac{d\phi(x)}{dx}=2\,\frac{e^x([x])!}{x^{[x]}}\cdot\left\{\frac{e^x[x]}{x^{[x]}}-\frac{e^x.[x].([x])!}{a^{[x]+1}}\right\}-2\pi$$

$$=2\left\{\frac{e^x([x])!}{x^{[x]}}\right\}^2\left(1-\frac{[x]}{x}\right)-2\pi.$$

Now when n is very large $1-\frac{[x]}{x}\to 0$, but $x-[x]$ can have any value between 0 and 1.

Thus $\frac{d\phi}{dx}=2(2\pi x+k)\left\{\frac{x-[x]}{x}\right\}-2\pi=4\pi(x-[x])-2\pi$, in the limit

and $\frac{d\phi}{dx}=0$ for $x=[x]+\frac{1}{2}$; and this gives the minimum value of $\phi(x)$ when x is large.

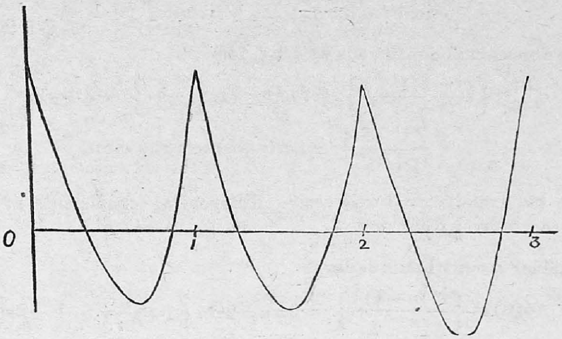
We evalute therefore

$$\left\{\frac{e^{n+\frac{1}{2}}.n\,!}{(n+\frac{1}{2})^n}\right\}^2-2\pi(n+\tfrac{1}{2})$$

$$=\frac{e.n^{2n}}{(n+\frac{1}{2})^{2n}}.2\pi n\left(1+\frac{1}{6n}+\ldots\right)-2\pi\left(n+\frac{1}{2}\right)$$

$$=2\pi n\left(1+\frac{1}{4n}+\ldots\right)\left(1+\frac{1}{6n}+\right)-2\pi\left(n+\frac{1}{2}\right)$$

$$=-\frac{\pi}{6}.$$



[A graph of the function between $x=0$ and $x=3$ is attached.]

Question 746.

(LAKSHMI SHANKAR N. BHATT) :—Prove that, n being an integer not less than 2,

$$\left\{\frac{1}{n-1}+\frac{1}{2}.\frac{1}{3n-1}+\frac{1\cdot3}{2\cdot4}.\frac{1}{5n-1}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}.\frac{1}{7n-1}\ldots\text{ad. inf.}\right\}\times$$

$$\left\{\frac{1}{n+1}+\frac{1}{2}.\frac{1}{3n+1}+\frac{1\cdot3}{2\cdot4}.\frac{1}{5n+1}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}.\frac{1}{7n+1}\ldots\text{ad. inf.}\right\}$$

$$=\frac{2}{(n-1)(n+1)}+\frac{2}{(3n-1)(3n+1)}+\frac{2}{(5n-1)(5n+1)}\ldots$$

Solution by S. V. Venkatachala Iyer and K. Appukuttan Erady.

Now since

$$\frac{x^{n-1}}{n-1}+\frac{1}{2}\frac{x^{3n-1}}{3n-1}+\frac{1\cdot3}{2\cdot4}\frac{x^{5n-1}}{5n-1}\ldots\ldots\text{ad. inf.}$$

$$=\int x^{n-2}(1-x^{2n})^{-\frac{1}{2}}dx,$$

and $$\frac{x^{n+1}}{n+1}+\frac{1}{2}\cdot\frac{x^{3n+1}}{3n+1}+\frac{1\cdot 3}{2\cdot 4}\frac{x^{5n+1}}{5n+1}+\ldots\ldots\text{ad. inf.}$$
$$=\int x^n(1-x^{2n})^{-\frac{1}{2}}dx,$$

and since both the series are convergent for values of x lying between the limits 0 and 1 (inclusive), we have, for integral values of $n \not< 2$, the left hand side of the given identity

$$=\int_0^1 x^{n-2}(1-x^{2n})^{-\frac{1}{2}}dx\times\int_0^1 x^n(1-x^{2n})^{-\frac{1}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{n-2}{n}}.\ \sec\theta.\ \frac{1}{n}(\sin\theta)^{\frac{1}{n}-1}\cos\theta\, d\theta$$

$$\times\int_0^{\frac{\pi}{2}}\sin\theta.\ \sec\theta.\ \frac{1}{n}(\sin\theta)^{\frac{1}{n}-1}\cos\theta\, d\theta$$

[by the substitution $x=\sin\theta$]

$$=\frac{1}{n^2}.\int_0^{\frac{\pi}{2}}(\sin\theta)^{-\frac{1}{n}}d\ \theta.\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{1}{n}}d\theta$$

$$=\frac{\Gamma\left(\frac{n-1}{2n}\right).\Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{2n-1}{2n}\right).\Gamma\left(\frac{2n+1}{2n}\right)}=\frac{\pi}{2n}\tan\frac{\pi}{2n}, \qquad \ldots \quad (1)$$

since $$\Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}.$$

But since $\cos\theta=\left\{1-\frac{4\theta^2}{\pi^2}\right\}\left\{1-\frac{4\theta^2}{3^2\pi^2}\right\}\ldots\ldots,$

$$\cos\frac{\theta}{2}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{5^2\pi^2}\right)\ldots\ldots$$
$$=\frac{\pi-\theta}{\pi}.\frac{\pi+\theta}{\pi}.\frac{3\pi-\theta}{3\pi}.\frac{3\pi+\theta}{3\pi}\ldots\ldots$$

Taking logarithms and differentiating

$$\tfrac{1}{2}\tan\frac{\theta}{2}.=\left[\frac{\pi}{\pi-\theta}.\frac{1}{\pi}-\frac{\pi}{\pi+\theta}.\frac{1}{\pi}\right]$$
$$+\left[\frac{3\pi}{3\pi-\theta}.\frac{1}{3\pi}-\frac{3\pi}{3\pi+1}.\frac{1}{3\pi}\right]\ldots$$
$$=\left\{\frac{1}{\pi-\theta}-\frac{1}{\pi+\theta}\right\}+\left\{\frac{1}{3\pi-\theta}-\frac{1}{3\pi+\theta}\right\}\ldots$$
$$=\frac{2\theta}{(\pi-\theta)(\pi+\theta)}+\frac{2\theta}{(3\pi-\theta)(3\pi+\theta)}\ldots\ldots$$

Putting $\theta=\frac{\pi}{n}$, we have

$$\frac{1}{2}\tan\frac{\pi}{2n}=\frac{\frac{2\pi}{n}}{\left(\pi-\frac{\pi}{n}\right)\left(\pi+\frac{\pi}{n}\right)}+\frac{\frac{2\pi}{n}}{\left(3\pi-\frac{\pi}{n}\right)\left(3\pi+\frac{\pi}{n}\right)}+\ldots\ldots(2)$$

$$\therefore\ \frac{\pi}{2n}\tan\frac{\pi}{2n}=\frac{2}{(n-1)(n+1)}+\frac{2}{(3n-1)(3n+1)}+\ldots\ldots$$

From (1) and (2), the given result follows.

Question 749.

(S. Krishnaswami Aiyangar) :—If ρ be the radius of curvature of the curve $r^m=a^m\sin m\theta$, ρ_k the radius of curvature at the corresponding point of its kth negative pedal with regard to the pole, show that $(1-mk+m)^{m-1}a^{mk}\rho_k^{m-1}=(1-mk)^{m-1}\{(m+1)\rho\}^{mk+m-1}$.

Solution by S. V. Venkataraya Sastri, M.A., L.T., M. M. Thomas and others.

The kth negative pedal is $r^n=a^n\sin n\theta$, where $n=m/(1-mk)$.

The point on the kth negative pedal corresponding to (r,θ) in the original curve is $[r^{1-mk}a^{mk},(mk+k-1)\theta.]$

We know $$\rho=\frac{a^m}{(m+1)r^{m-1}}$$

$$\therefore\qquad \rho_k=\frac{a^n}{(n+1)r^{n-1}}.$$

Hence, after a slight simplification, the required result follows.

Additional solutions by K. Srinivasan, K. B. Madhava and S. V. Venkatachala Iyer.

Question 760.

(K. Appukuttan Erady, M. A.) :—If

$$f(n)\equiv\frac{1}{n}+\frac{1}{n(n+1)}+\frac{1}{n(n+1)(n+2)}+\cdots,$$

show that

$$f(1)+\frac{f(2)}{1!}+\frac{f(3)}{2!}+\ldots\ldots=e.$$

Solution (1) *by M. K. Kewalramani, M.A., G. V. M. Gaitonde and others;* (2) *by M. Viraswamaiya, Durai Rajan, K. K. Ranganatha Aiyar and Lakshmi Sankar N. Bhatt B.A.* (3) *by A. Narasinga Rao;* *and* (4) *by K. B. Madhava.*

(1) Let $f(1)=t_1 : \frac{f(2)}{1\,!}=t_2 : \frac{f(3)}{2\,!}=t_3$, etc. and S their sum.

$$t_r=\frac{f(r)}{(r-1)\,!}$$

$$=\frac{1}{(r-1)\,!}\left\{\frac{1}{r}+\frac{1}{r(r+1)}+\frac{1}{r(r+1)(r+2)}+\dots\right\}$$

$$=\frac{1}{r\,!}+\frac{1}{(r+1)\,!}+\frac{1}{(r+2)\,!}+\frac{1}{(r+3)\,!}+\dots$$

$$\therefore \quad t_1=\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\dots$$

$$t_2=\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\frac{1}{5\,!}+\dots$$

$$t_3=\frac{1}{3\,!}+\frac{1}{4\,!}+\frac{1}{5\,!}+\frac{1}{6\,!}+\dots$$

$$t_4=\frac{1}{4\,!}+\frac{1}{5\,!}+\frac{1}{6\,!}+\frac{1}{7\,!}+\dots$$

From the above it is clear that the sum S contains:

$$\text{one } \frac{1}{1!}\text{; two } \frac{1}{2\,!}\text{; three } \frac{1}{3\,!}\text{; four } \frac{4}{4\,!}\text{; five } \frac{5}{5\,!}$$

and so on.

$$S=\frac{1}{1!}+2\left(\frac{1}{2\,!}\right)+3\left(\frac{1}{3\,!}\right)+4\left(\frac{1}{4\,!}\right)+5\left(\frac{1}{5\,!}\right)+6\left(\frac{1}{6\,!}\right)$$

$$+\dots\dots\dots\dots+r\left(\frac{1}{r\,!}\right)+(r+1)\ \frac{1}{(r+1)!}+\dots$$

$$=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\frac{1}{5\,!}+\dots\dots\dots\dots\dots$$

$$=e.$$

(2) Now
$$nf(n)=1+\frac{1}{(n+1)}+\frac{1}{(n+1)(n+2)}+\dots$$

$$=1+f(n+1)$$

$$\therefore \quad nf(n)-f(n+1)=1.$$

In this give to n values 1, 2, 3,...and we get a series of identities

$$f(1)-f(2)=1$$

$$2f(2)-f(3)=1$$

$$3f(3)-f(4)=1, \textit{etc.}$$

or which is the same thing as

$$f\ (1)-f(2)=1$$
$$2f\ (2)-f(3)=1$$
$$\frac{3}{2!}f\ (3)-\frac{1}{2!}f\ (4)=\frac{1}{2!}$$
$$\frac{4}{3!}\ f\ (4)-\frac{1}{3!}\ f\ (5)=\frac{1}{3!}$$

and so on, adding the various identities, we get

$$f(1)+\frac{f(2)}{1!}+\frac{f(3)}{2!}+\frac{f(4)}{3!}+\frac{f(5)}{1!}+.$$
$$=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1!}{4}+\ldots$$

(3) $$f(n)=n^{(-1)}+n^{(-2)}+n^{(-3)}\ldots\ldots$$
$$=\left(1-\Delta+\frac{\Delta^2}{2!}-\frac{\Delta^3}{3!}\ldots\right)\frac{1}{n}$$
$$=e^{-\Delta}\left(\frac{1}{n}\right)$$

Hence $$\frac{e}{n}=e^{1+\Delta}.e^{-\Delta}\frac{1}{n}=e^{\mathrm{E}}f(n)$$
$$=f(n)+\frac{f(n+1)}{1!}+\frac{f(n+2)}{2!}+\ldots \qquad \ldots(\mathrm{i})$$

The given result follows on putting $n=1$.

(4) We have Prym's Identity (see p. 149, vol. vii., or p. 17, vol. viii, J . I. M. S.)

$$f(n)\equiv e\left[\frac{1}{n}-\frac{1}{1!}\ \frac{1}{n+1}+\frac{1}{2!}\ \frac{1}{n+2}-\frac{1}{3!}\ \frac{1}{n+3}+\ldots\right]$$

from which by successive substitutions we have

$$f(1)+\frac{f(2)}{1!}+\frac{f(3)}{2!}+\ldots$$
$$=e\left[1-\frac{1}{1!}\frac{1}{2}+\frac{1}{2!}\frac{1}{3}-\frac{1}{3!}\frac{1}{4}+\ldots\right.$$
$$+\frac{1}{1!}\left\{\frac{1}{2}-\frac{1}{1!}\frac{1}{3}+\frac{1}{2!}\ \frac{1}{4}-\ldots\right.$$
$$\left.+\frac{1}{2!}\left\{\frac{1}{3}-\frac{1}{1!}\frac{1}{4}+\ldots\right.\right]$$
$$=e,$$

since $\frac{1}{n!}-\frac{1}{1!}\ \frac{1}{n-1!}+\frac{1}{2!}\ \frac{1}{n-2!}-\ldots$to $(n+1)$ terms

$$=\text{coefficient of } x^n \text{ in } e^x\times e^{-x}=0.$$

Question 767.

(S. KRISHNASWAMI IYENGAR) :—Show that

$$1-\frac{1}{2^2.5}+\frac{1}{2^4.9}\cdots(-)^n\frac{1}{2^{2n}(4n+1)}=\tfrac{1}{4}\log 5+\tfrac{1}{2}\tan^{-1}2.$$

Solution by Prof. K. J. Sanjana and K. B. Madhava.

More generally,

let
$$f(x)=x-\frac{1}{2^2}\frac{x^5}{5}+\frac{1}{2^4}\frac{x^9}{9}-\ldots ;$$

then
$$f'(x)=1-\frac{1}{2^2}x^4+\frac{1}{2^4}x^8-\ldots=\frac{4}{4+x^4}.$$

Hence $f(x)=C+\int\frac{4dx}{4+x^4}$.

Now $C=0$ evidently, as $x\not>1$;

and
$$\int\frac{4dx}{4+x^4}=\sqrt{2}\int\frac{dy}{1+y^4},\text{ putting } x=y\sqrt{2},$$

$$=\frac{\sqrt{2}}{4}\sqrt{2}\left\{\tanh^{-1}\frac{y\sqrt{2}}{1+y^2}+\tan^{-1}\frac{y\sqrt{2}}{1-y^2}\right\}$$

$$=\tfrac{1}{2}\left(\tanh^{-1}\frac{2x}{2+x^2}+\tan^{-1}\frac{2x}{2-x^2}\right).$$

Putting $x=1$, we get

$$\text{the given series}=\tfrac{1}{2}(\tanh^{-2}\tfrac{2}{3}+\tan^{-1}2)$$
$$=\tfrac{1}{4}\log 5+\tfrac{1}{2}\tan^{-1}2.$$

Similar solutions by M. K. Kewalramani, S. V. Venkatachala Aiyar and K. Appukuttan Erady.

Question 768.

(S. RAMANUJAN) :—If $\psi(x)=\frac{x+2}{x^2+x+1}$, show that

$$\text{(i)}\quad \tfrac{1}{3}\psi(x^{\frac{1}{3}})+\tfrac{1}{9}\psi(x^{\frac{1}{9}})+\tfrac{1}{27}\psi(x^{\frac{1}{27}})+\ldots=\frac{1}{\log x}+\frac{1}{1-x}$$

for all positive values of x ; and

$$\text{(ii)}\quad \tfrac{1}{3}\psi(x^{\frac{1}{3}})+\tfrac{1}{9}\psi(x^{\frac{1}{9}})+\tfrac{1}{27}\psi(x^{\frac{1}{27}})+\ldots=-\frac{1}{1-x}$$

for all negative values of x.

Solution by K. J. Sanjana and N. Durairajan.

(i) Since $$\frac{x+2}{x^2+x+1}=\frac{1}{x-1}-\frac{3}{x^3-1},$$

$$\text{the given series}=\left(\frac{1}{3}\frac{1}{x^{\frac{1}{3}}-1}-\frac{1}{x-1}\right)+\left(\frac{1}{9}\frac{1}{x^{\frac{1}{9}}-1}-\frac{1}{3}\frac{1}{x^{\frac{1}{3}}-1}\right)$$

$$+\left(\frac{1}{27}\cdot\frac{1}{x^{\frac{1}{27}}-1}-\frac{1}{9}\frac{1}{x^{\frac{1}{9}}-1}\right)+\ldots$$

$$=-\frac{1}{x-1}+\frac{1}{3^n}\cdot\frac{1}{x^{3^{-n}}-1}, \text{ for } n \text{ terms.}$$

When $n\to\infty$, the limit of the second part is $\frac{1}{\log x}$.

(ii) Put $y=-x$, so that y is positive. Then

$$\psi(x)=\frac{2-y}{1-y+y^2}=\frac{3}{1+y^3}-\frac{1}{1+y}.$$

Proceeding as above we shall get the sum of n terms to be

$$\frac{1}{1+y}-\frac{1}{3^n}\frac{1}{1+y^{3^{-n}}};$$

the limit of the second term for $n\to\infty$ being zero,

$$\text{the required sum}=\frac{1}{1-x}.$$

Similarly, the limit of

$$\sum_1^{\infty}\frac{1}{r^n}\psi\left(x^{r^{-n}}\right)=\frac{1}{\log x}-\frac{1}{x-1} \text{ or } \frac{-1}{x-1},$$

if $\psi(x)$ stands for $\dfrac{(r-1)+(r-2)+(r-3)x^2+\ldots x^{r-2}}{1+x+x^2+\ldots x^{r-1}}$.

Question 775.

(K. Srinivasan):—Prove geometrically $1+\text{dn } 2z=\dfrac{2\text{dn}^2 z}{1-k^2\text{sn}^4 z}$

Solution by F. H. V. Gulasekaram, B.A.,

Take any four points M, m, n N in order on a st. line, and on MN, mn as diameters describe circles. Let the centres be O, O_1; the radii R, r; and let $O\,O_1=\delta$.

Draw chords of the outer circle MA, AD, MB to touch the inner. Draw another circle having its centre on MN and touching AB and MD.

Let the points of contact be as shewn in the figure.

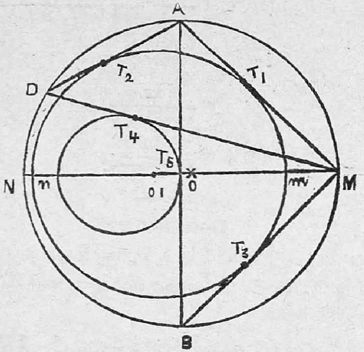
It is evident that T_5 is the mid-point of AB and that it lies on MN.

As in Dixon: *Elliptic Functions*, §§ 105—107, let us suppose that the point M on the outer circle corresponds to the argument zero, A to the argument u; hence D corresponds to the argument $2u$ and B to $-u$.

Also,

$$\mathrm{dn}(u,k)=\frac{AT_1}{MT_1}, \text{ where } k^2=\frac{4R\delta}{(R+\delta)^2-r^2};$$

and $$\hat{MNA}=\theta=\tfrac{1}{2}\text{ am. } u,\ \hat{MND}=\phi=\tfrac{1}{2}\text{ am. } 2u$$



Hence we have

$$\frac{OA^2}{OM^2}=\frac{R^2+\delta^2+2R\delta\cos 2\theta}{OM^2}$$

$$=1-\frac{4R\delta\sin^2\theta}{OM^2}$$

$$=1-\frac{4R\delta}{(R+\delta)^2-r^2}\cdot\frac{MT^2}{OM^2}$$

$$=1-k^2\sin^4\text{am. }u$$

$$=1-k^2\,\mathrm{sn}^4\,u.$$

Again since it can be proved that the three circles in the fig. have the same radical axis [*Vide*: Ex. 45, p. 417. Nixon: *Euc. Revised*], the elliptic functions have the same modulus whether the inner circle be the one with the centre O_1, or the other touching AB, MD.

Again from the circle touching AB, MD, we have

$$1+\text{dn}\,2u=\frac{\text{MD}}{\text{MT}_4}$$

and

$$\text{dn}\,u=\frac{\text{AT}_5}{\text{MT}_4}$$

$$\therefore\quad \frac{1+\text{dn}\,2u}{\text{dn}\,u}=\frac{\text{MD}}{\text{AT}_5},$$

$$=\frac{2\text{R}\sin\phi}{2\text{R}\sin\theta\cos\theta}$$

$$=\frac{\sin 2\,\text{O}_1\text{AT}_1}{\sin\theta\cos\theta}.$$

$$=\frac{2.\text{OT}_1}{\text{OA}}\cdot\frac{\text{AT}_1}{\text{OA}}\cdot\frac{\text{OM}}{\text{OT}_1}\cdot\frac{\text{OM}}{\text{MT}_1}$$

$$=\frac{2.\text{AT}_1}{\text{MT}_1}\cdot\frac{\text{OM}^2}{\text{OA}^2}.$$

$$=\frac{2\,\text{dn}\,u}{1-k^2\text{sn}^4u}$$

Hence

$$1+\text{dn}\,2u=\frac{2\,\text{dn}^2\,u}{1-k^2\,\text{sn}^4u}.$$

Question 776.

(K. SRINIVASAN) :—Expand in a Fourier series $\text{cn}^3 z$.

Solution by F. H. V. Gulasekaram, B.A. and K. B. Madhava.

Method 1.—We have

$$\frac{d}{dz}(\text{sn}\,z\,\text{dn}\,z)=2k^2\,\text{cn}^3 z+\text{cn}\,z\,(1-2\,k^2)$$

Hence

$$2\,k^2\,\text{cn}^3 z=-\frac{d^2}{dz^2}\,\text{cn}\,z-\text{cn}\,z\,(1-2\,k^2)\qquad\ldots\qquad\ldots\qquad(\text{i})$$

Now

$$\text{cn}\,z=\frac{2\pi}{\text{K}k}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}\cos(2n+1)x}{1+q^{2n+1}},$$

valid throughout the strip $|\text{I}(x)|<\frac{1}{2}\pi\,\text{I}(\text{J})$,

where

$$z=\frac{2\text{K}x}{\pi},\ q=e^{\pi i \text{J}}\ ;\ \text{J}=\frac{i\text{K}'}{\text{K}}.$$

(*Vide* : *Ex.* 1, *p.* 504, Whittaker's *Modern Analysis.*)

Hence from (i)

$$\text{cn}^3 z=\sum_{u=0}^{\infty}\left[\frac{(2n+1)^2}{2k^3}\left(\frac{\pi}{2\text{K}}\right)^2-\frac{(1-2k^2)}{2k^3}\right]\frac{2\pi q^{n+\frac{1}{2}}\cos(2n+1)x}{\text{K}\,(1+q^{2n+1})}$$

valid throughout the strip $|\text{I}(x)|<\frac{1}{2}\pi\,\text{I}(\text{J})$.

(*Vide*: Whittaker *Ex.* 57, p. 528, for the analogous expansion of $\text{sn}^3 z$.

Method 2.—If $z = \frac{2Kx}{\pi}$, $\text{cn}^3 z$ is an even periodic function of x (with period 2π) which obviously satisfies Dirichlet's conditions for real values of x and therefore

$$\text{cn}^8 z = \frac{1}{2}\, a_o + \sum_{n=1}^{\infty} a_n \cos nx,$$

the expansion being valid for all real values of x.

The co-efficients a_n are given by the formula

$$\pi a_n = \int_{-\pi}^{+\pi} \text{cn}^8 z \exp\,(nix)\, dx.$$

To evaluate this integral consider $\int \text{cn}^3 z. \exp\,(nix)\, dx$, taken round the parallelogram whose corners are

$$-\pi,\ \pi,\ \pi\,J, -2\pi + \pi\,J.$$

From the periodic properties of $\text{cn}^3 z$ and $\exp\,(nix)$, we see that

$$\int_{\pi}^{\pi J} \text{ cancels } \int_{-2\pi+\pi J}^{-\pi}\ ;$$

again, $-\pi + \frac{1}{2}\pi J$ and $\frac{1}{2}\,\pi J$ are the only poles of the integrand (*qua* function of x) inside the contour.

Again, $\text{cn}^8(z+iK').\ \exp\{\, ni\,(x + \frac{1}{2}\,\pi J)\,\}$

$$= \frac{+i}{k^3}\ \text{ds}^3 z \exp\,(nix)\, q^{\frac{1}{2}n},\ (\text{where } q = e^{\pi i J})$$

$$= \frac{iq^{\frac{1}{2}n}}{k^3 z^3}\left[1 + \tfrac{1}{2}(1-2k^2)z^2 + \ldots\right]\left[1 + nix - \frac{n^2x^2}{2} + \ldots\right];$$

so that, the residue of the integrand at the pole $\frac{1}{2}\pi J$

$$= iq^{\frac{1}{2}n}\left(\frac{\pi}{2K}\right)\left[\frac{1-2k^2}{2k^3} - \frac{n^2}{2k^3}\left(\frac{\pi}{2K}\right)^2\right];$$

and the residue at the pole $-\pi + \frac{1}{2}\pi J$

$$= -i(-1)^n\, q^{\frac{1}{2}n}\left(\frac{\pi}{2K}\right)\left[\frac{1-2k^2}{2k^8} - \frac{n^2}{2k^3}\left(\frac{\pi}{2K}\right)^2\right].$$

Hence

$$\left\{\int_{-\pi}^{\pi} - \int_{-2\pi+\pi J}^{\pi J}\right\} \text{cn}^3 z.\ \exp\,(nix)\, dx$$

$$= \frac{\pi^2}{K}\left[1 - (-1)^n\right]\left[\frac{n^2}{2k^8}\left(\frac{\pi}{2K}\right)^2 - \frac{1-2k^2}{2k^8}\right] q^{\frac{1}{2}n}.$$

Writing $(x-\pi+\pi)$ J for x in the second integral, we have

$$\{1-(-1)^n q^n\}\int_{-\pi}^{\pi} \text{cn}^3 z.\ \exp(nix)\,dx$$
$$=\frac{\pi^2}{K}\Big[1-(-1)^n\Big]\Big[\frac{n^2}{2k^3}\Big(\frac{\pi}{2K}\Big)^2-\frac{1-2k^2}{2k^3}\Big]\,q^{\frac{1}{2}n}$$

When n is *even*, $a_n=0$; but when n is *odd*

$$a_n=\frac{2\pi}{K}\Big[\frac{n^2}{2^3k}\Big(\frac{\pi}{2K}\Big)^2-\frac{1-2k^2}{2k^3}\Big]\frac{q^{\frac{1}{2}n}}{1+q^n}\,;$$

and a_o is easily seen to be zero.

Hence, we get the expansion already obtained by *Method* 1.

Note :—The Fourier Series for $(\text{sn}\ u)^m$, $(\text{cn}\ u)^m$, $(\text{dn}\ u)^m$ and their reciprocals can be found as follows :

We have, when m is any positive integer,(Dixon's *Elliptic Functions* §§ 56, 59.)

(A) $$\Big\{\frac{1}{m-2}\frac{d^2}{du^2}(s^{m-2})=\frac{d}{du}(s^{m-3}\,cd)\Big\}$$
$$=(m-1)k^2s^m-(m-2)(1+k^2)s^{m-2}+(m-3)s^{m-4}$$

(B) $$\Big\{\frac{1}{m-2}\frac{d^2}{du^2}(s^{-m+2})=-\frac{d}{du}(s^{-m+1}\,cd)\Big\}$$
$$=(m-1)s^{-m}-(m-2)(1+k^2)s^{-m+2}+(m-3)k^2s^{-m+4}$$

where s, c, d stand for sn u, cn u, dn u respectively.

Hence $(\text{cn}\ u)^m$ can be expanded in a Fourier Series by differentiation as follows :—

(1) When m is odd and greater than one, the series for $(\text{sn}\ u)^m$ is obtained by differentiation from the series for sn u by means of the reduction formula (A).

(2) When m is even and greater than 2, the series for $(\text{sn}\ u)^m$ is obtained by differentiation from the series for $\text{sn}^2 u$ by means of the reduction formula (A).

(3) When m is odd and greater than 1, the series for $(\text{ns}\ u)^m$ is obtained by differentiation from the series for ns u by means of the reduction formula (B),

(4) When m is even and greater than 2, the series for $(\text{ns}\ u)^m$ is obtained by differentiation from the series for $\text{us}^2 u$ by means of the reduction formula (B).

Hence the expansion for $(\text{sn } u)^m$ is made to depend on either that of sn u or $\text{sn}^2 u$, and the expansion of $(\text{ns } u)^n$ is made to depend on either that of ns u or $\text{ns}^2 u$.

Again since the expansions for ns u and $\text{ns}^2 u$ are deducible from those of sn u and $\text{sn}^2 u$ by writing $u+i\,\text{K}'$ for u in the latter, we see that the expansion of $(\text{sn } u)^m$ may be made to depend on that of sn u when m is odd and on that of $\text{sn}^2 n$ when m is even, m being positive or negative

By constructing similar reduction formulae for $(\text{cn } u)^m$, $(\text{dn } u)^m$, *etc.* the Fourier series for $(\text{cn } u)^m$, *etc.*, may be obtained.

Question 785.

(S. RAMANUJAN) :—Show that

$$\sqrt[3]{\{3(\sqrt[3]{a^3+b^3}-a)(\sqrt[3]{a^3+b^3}-b)\}}=\sqrt[3]{\{(a+b)^2\}}-\sqrt[3]{(a^2-ab+b^2)}.$$

This is analogous to

$$\sqrt{\{2(\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}-b)\}}=(a+b)-\sqrt{(a^2+b^2)}.$$

Solution by K. K. Ranganatha Aiyar, R. D. Karve, G. A. Kamtekar, L. N. Datta and L. N. Subramanyam.

In the identity

$(a+b-r)^3 = (a+b)^3-r^3-3r(a+b)^2+3r^2(a+b)$, put $a^3+b^3=r^3$; we have

$$(a+b-r)^3=3ab(a+b)-3r(a+b)^2+3r^2(a+b)$$
$$=3(a+b)(r-a)(r-b)$$

$$\therefore \qquad \sqrt[3]{3(r-a)(r-b)}=(a+b)^{\frac{2}{3}}-\left(\frac{a^3+b^3}{a+b}\right)^{\frac{1}{3}},$$

which is equivalent to the result required.

Again, in $(a+b-r)^2=2ab-2r(a+b)+2r^2$,

put $r^2=a^2+b^2$, and we get

$$\therefore \qquad \sqrt{2(r-a)(r-b)}=a+b-r=a+b-\sqrt{a^2+b^2}$$

Question 787.

(M. K. KEWALRAMANI, M. A.) :—Prove that, if a be not an integer,

$$\frac{\pi}{2}\frac{f(x+ah)-f(x-ah)}{\sin a\pi}=\frac{f(x+h)-f(x-h)}{1^2-a^2}$$

$$2\frac{f(x+2h)-f(x+2h)}{2^2-a^2}+3\frac{f(x+3h)-f(x-3h)}{3^2-a^2}-\ldots$$

Solution by V. M. Gaitonde, K. Âppukuttan Erady and K. K. Ranganatha Aiyar.

We can, when y lies between $\pm\pi$, by using the Fourier's Theorem, shew that

$$\sin ay = \frac{2}{\pi}\sin a\pi\left(\frac{\sin y}{1-a^2}-\frac{2\sin 2y}{2^2-a^2}+\frac{3\sin 3y}{3^2-a^2}-\dots\right),$$

[Williamson : *Int. Calc.*, p. 400, Ex. 10.]

Hence we get

$$\frac{\pi}{2}\cdot\frac{e^{ayi}-e^{-ayi}}{\sin a\pi}=\frac{e^{yi}-e^{-yi}}{1-a^2}-2\frac{e^{2yi}-e^{-2yi}}{2^2-a^2}+\dots$$

Putting $e^{yi}\equiv e^{h\frac{d}{dx}}$ and applying each side to the function $f(x)$, we get at once

$$\frac{\pi}{2}\cdot\frac{f(x+ah)-f(x-ah)}{\sin a\pi}=\frac{f(x+h)-f(x-h)}{1^2-a^2}$$
$$-2\frac{f(x+2h)-f(x-2h)}{2^2-a^2}+3\frac{f(x+3h)-f(x-3h)}{3^2-a^2}-\dots$$

Addition Solution by L. N. Bhatt, B.A.

Question 788.

(E. H. NEVILLE):—From the point (u,v) can be drawn four normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; four circles are drawn, each through the feet of three of these normals; shew that the sum of the squares of the radii of these circles is given by

$$2a^2b^2\Sigma r^2=(a^2+b^2)^3-(a^2-b^2)(a^2u^2-b^2v^2).$$

Solution by F. H. V, Gulasekaram, B.A., K. B. Madhava, V. M. Gaitonde, K. K. Ranganatha Aiyar, and others.

Let $\alpha,\beta,\gamma,\delta$ be the eccentric angles of the feet of the 4 normals.

Then $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta=\frac{2au}{a^2-b^2}$ ... ... (i)

$\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=\frac{-2bv}{a^2-b^2}$ ... ... (ii)

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=2+\frac{2(a^2u^2-b^2v^2)}{(a^2-b^2)^2}$ ... (iii)

Consider the circle through the points α,β,γ. This meets the circle again at the point $(\pi+\delta)$; so that, after C. Smith's *Conic Sections*, p. 169,

$$\cos(\alpha+\beta+\gamma)=-\cos\delta$$
$$\sin(\alpha+\beta+\gamma)=\sin\delta.$$

If (x_1, y_1) be the centre, r_1 the radius of the circle

$$x_1=\frac{a^2-b^2}{4a}\left[\cos\alpha+\cos\beta+\cos\gamma-\cos\delta\right]$$

$$=\frac{u}{2}-\frac{a^2-b^2}{2a}\cos\delta. \qquad \text{[from (i)]}$$

$$y_1=-\frac{a^2-b^2}{4b}\left[\sin\alpha+\sin\beta+\sin\gamma-\sin\delta\right]$$

$$=\frac{v}{2}+\frac{a^2-b^2}{2b}\sin\delta. \qquad \text{[from (ii)]}$$

$$r_1^2=(x_1+a\cos\delta)^2+(y^2_1+b\sin\delta)^2$$

$$=\left(\frac{a^2+b^2}{2a}\cos\delta+\frac{u}{2}\right)+\left(\frac{a^2+b^2}{2b}\sin\delta+\frac{v}{2}\right)^2$$

$$=\frac{(a^2+b^2)^2}{4a^2}\left[\cos\delta+\frac{au}{a^2+b^2}\right]^2+\frac{(a^2+b)^2}{4b^2}\left[\sin\delta+\frac{bv}{a^2+b^2}\right]^2.$$

Hence taking the radii of the four circles in question, and using the results (i), (ii), (iii) above, we have

$$4a^2b^2\Sigma r^2=b^2(a^2+b^2)^2\left[2+\frac{2(a^2u^2-b^2v^2)}{(a^2-b^2)^2}\right]$$

$$+a^2(a^2+b^2)^2\left[2-2\frac{(a^2u^2-b^2v^2)}{(a^2-b^2)^2}\right]$$

$$+4a^2b^2(u^2+v^2)+4\frac{a^2+b^2}{a^2-b^2}a^2b^2(u^2-v^2)$$

$$=2(a^2+b^2)^3-2(a^2-b^2)(a^2u^2-b^2v^2), \text{ after reduction.}$$

Hence the result.

Additional Solution by K. Appukuttan Erady.

Question 789.

(K. J. SANJANA) :—P, Q, R, S are four co-normal points on an ellipse whose centre is C and axes $2a$ and $2b$, O being the point of concurrence of the normals. If x_r, y_r ($r=1, 2, 3, 4$) denote the centres of the circles QRS, PRS, PQS, PQR respectively, prove that

$$\Sigma x=\text{abscissa of O}, \ \Sigma y=\text{ordinate of O}$$

and, that each centre lies on an ellipse whose centre is at the mid-point of CO and whose axes are

$$\frac{a^2-b^2}{2a}, \frac{a^2-b^2}{2b}. \text{ [Suggested by Q 788].}$$

Solution by F. H. V. Gulasekaram, B.A., S. V. Venkatachala Aiyar, K. B. Madhava and M. M. Thomas.

Let $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four points P, Q R, S; Let O be the point $(u, v.)$

Then if (x_4, y_4) be the centre of the circle PQR,

$$x_4 = \frac{u}{2} - \frac{a^2-b^2}{2b}\cos\delta \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$y_4 = \frac{v}{2} + \frac{a^2-b^2}{2b}\sin\delta \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

[*Vide: Solution to Q.* 788.]

$$\therefore \quad \Sigma x = 2u - \frac{a^2-b^2}{2a}\Sigma\cos\alpha$$

$$= 2u - \frac{a^2-b^2}{2a}\cdot\frac{2au}{a^2-b^2}$$

$$= u, \text{ the abscissa of O.}$$

Similarly, $\Sigma\, y = v$

$=$ the ordinate of O.

From (i) and (ii), it is evident that each centre lies on an ellipse whose centre is at the middle point of CO and whose axes are

$$\frac{a^2-b^2}{2a},\ \frac{a^2-b^2}{2b}.$$

Question 794.

(Martyn M. Thomas, M.A.) :—The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ moves in such a manner that the centroid of the triangle ABC always lies on the surface $x^{-2}+y^{-2}+z^{-2}=9$, A, B, C being the points where the plane meets the rectangular axes of co-ordinates. Prove that the nine-points centre of ABC will lie on the surface

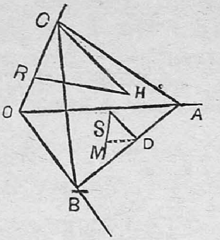
$$x\sqrt{4x^2-1}+y\sqrt{4y^2-1}+z\sqrt{4z^2-1}+2(x^2+y^2+z^2)=1.$$

Solution by K. J. Sanjana, M.A. and K. B. Madhava.

The co-ordinates of the centroid are evidently $\frac{1}{3}a$, $\frac{1}{3}b$, $\frac{1}{3}c$; since it lies on the surface,

$$\left(\frac{a}{3}\right)^{-2}+\left(\frac{b}{3}\right)^{-2}+\left(\frac{c}{3}\right)^{-2}=\left(\frac{1}{3}\right)^{-2},$$

$$\therefore \quad \sum \frac{1}{a^2} = 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$



For the circumcentre S, we have

$$(x-a)^2+y^2+z^2 = x^2+(y-b)^2+z^2 = x^2+y^2+(z-c)^2,$$

along with the equation of the plane.

Thus we get z or $\mathrm{SM} = \dfrac{c^2-1}{2c}$; $y = \dfrac{b^2-1}{2b}$, $x = \dfrac{a^2-1}{2a}$.

If H is the orthocentre, and HR parallel to the xy—plane meets OC in R, the $\triangle$S HCR and SDM are similar, where SD is perpendicular to AB.

Hence, since CH = 2SD,

CR = 2SM ; and OR, or z for H, is

$$c - 2\frac{c^2-1}{2c} = \frac{1}{c}.$$

Thus the co-ordinates of the nine-points centre are given by

$$x = \tfrac{1}{2}\left\{\frac{1}{a} + \frac{a^2-1}{2a}\right\} = \frac{a^2+1}{4a};\ y = \frac{b^2+1}{4b},\ z = \frac{c^2+1}{4c}.$$

From these $a^2 - 4\,ax + 1 = 0$, &c. ;

$\therefore\ a = 2x \pm \sqrt{4x^2-1}$, and $\dfrac{1}{a} = 2x \pm \sqrt{4x^2-1}$, &c.

Substituting in condition (1) we get at once

$$\sum(8x^2-1) \mp \sum 4x\sqrt{4x^2-1} = 1,$$

or

$$\sum 2x^2 \mp \sum x\sqrt{4x^2-1} = 1,$$

which is the equation of the locus.

QUESTIONS FOR SOLUTION.

808. (K. J. SANJANA, M. A.) :—Prove that

$$\frac{1}{y(y+z)}+\frac{1}{(y+1)(y+z+1)}+\frac{1}{(y+2)(y+z+2)}+\ldots ad \text{ inf.}$$

$$=\frac{1}{y}-\frac{1}{2}\frac{z-1}{y(y+1)}+\frac{1}{3}\frac{(z-1)(z-2)}{y(y+1)(y+2)}-\frac{1}{4}\frac{(z-1)(z-2)(z-3)}{y(y+1)(y+2)(y+3)}+\ldots,$$

when y and z are positive rational numbers. Examine the identity for negative values of z.

809. (T. P. TRIVEDI, M. A., LL.B.) :—With the usual notation for an ellipsoid, and writing

$$\frac{1}{p}=\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2},$$

shew that

$$\int\frac{x+y+z}{p}\,dS=\frac{\pi}{16}abc\left\{(a+b+c)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right\}$$

the integral extending over that part of the ellipsoid for which x, y, z are all positive.

Write down also the value of

$$\int\frac{(x+y+z)^2}{p}dS \text{ in a symmetrical form.}$$

810. (T. P. TRIVEDI, M. A., LL.B.) :—Find the values (other than zero) which satisfy the equations :

$$x^2=y-z,\ y^2=z-x,\ z^2=x-y.$$

811. (C. KRISHNAMACHARY) :—Sum to n terms.

$$\text{(i)}\ 1+\frac{a^2}{(b+x)^2}\left(1+\frac{2x}{a+b}\right)+\frac{a^2(a+x)^2}{(b+x)^2(b+2x)^2}\left(1+\frac{4x}{a+b}\right)$$

$$+\frac{a^2(a+x)^2(a+2x)^2}{(b+x)^2(b+2x)^2(b+3x)^2}\left(1+\frac{6x}{a+b}\right)+\ldots$$

$$\text{(ii)}\ \frac{1}{y-x}+\frac{2(y+x)}{(y-x)(y-2x)}+\frac{3(y+x)(y+2x)}{(y-x)(y-2x)(y-3x)}+\ldots\ldots$$

812. (S. KRISHNASWAMI IYENGAR) :—If S_n represents the sum of the reciprocals of the first n odd numbers prove that

$$\sum_1^{\infty}\frac{S_n}{4^n(n\,!)^2}=\sum_1^{\infty}\frac{1}{n^2}.$$

813. (S. KRISHNASWAMI IYENGAR) :—Show how to find the sum of the series

$$1+\cos\alpha+\cos 2\alpha\,\frac{2^n}{2\,!}+\cos 3\alpha\,\frac{3^n}{3\,!}+\ldots\ldots$$

and prove that when $n=2$, the sum of the series is

$$\{\cos(\sin\alpha)(\cos\alpha+\cos 2\alpha)-\sin(\sin\alpha)(\sin\alpha+\sin 2\alpha)\}\,e^{\cos\alpha}.$$

814. (K. J. Sanjana, M. A.) :—A regular quindecagon is inscribed in the circle $x^2+y^2=1$, with one angular point on the axis of x. Prove that the special vertices of the polygon (*i.e.*, those not common to it and the inscribed regular pentagon or triangle) lie at the intersection of the circle with the two hyperbolas

$$2(x^2-y^2)-x\pm\sqrt{15}y-2=0.$$

Prove also that the special vertices of the similarly inscribed polygon of 21 sides lie on the two cubics

$$(4x-1)(x^2-y^2)-4x+2\sqrt{7}\,xy=0.$$

815. (Martyn M. Thomas, M. A.) :—If

$$\int\frac{dx}{e^{ax^2+bx+c}}=\sum\frac{x^n}{n}k_n,$$

prove that $2ak_{n-1}+bk_n+nk_{n+1}=0$.

816. (Martyn M. Thomas, M. A.) :—If k be the curvature of an ellipse, and

$$a_0=\frac{1}{2\,!}k,\ a_1=\frac{1}{3\,!}\frac{dk}{ds},\text{a}\ \ a_2=\frac{1}{4\,!}\frac{d^2k}{ds^2},$$

show that $a_0^{-\frac{8}{3}}(4a_0^4-5a_1^2+4a_0a_2)$ has the constant value $\left(\frac{2}{\alpha\beta}\right)^{\frac{2}{3}}$ at all points of the ellipse, α and β being the semi-axes.

817. (S. Malhari Rao.) :—If $^xC_9$ is equal to the continued product of three primes whose sum is 327, find x.

818. (S. Malhari Rao) .—Complete the following magic squares by inserting prime numbers in vacant cells :—

(*a*)

	1	
13		
		7

(*b*)

			47
		19	
	61		
53	83	1	43

(*c*)

		1		
		3		
103	97	449	43	37
		137		
		139		

819. (K. APPUKUTTAN ERADY) :—If ρ, ρ' be the radii of curvature at corresponding points of a curve and its α-evolute where α is a function of the arc measured from a fixed point on the curve, show

$$\rho'\left(\frac{1}{\rho}+\frac{d\alpha}{ds}\right)+\sin\alpha=\frac{d}{ds}\left\{\frac{\cos\alpha}{\frac{1}{\rho}+\frac{d\alpha}{ds}}\right\}.$$

820. (K. APPUKUTTAN ERADY) :—Prove that the equation

$$(r+r_1)(r-r_1)^2=4a(ar+r_1-\lambda rr_1)$$

in bipolar co-ordinates, represents the lines of force of a simple magnet, and show that the system of orthogonal curves is given by the equation

$$\frac{1}{r}-\frac{1}{r_1}=\mu.$$

821. (M. K. KEWALRAMANI) :—If through A, B, C lines AXY, BYZ, CZX are drawn so as to make the same angle θ with AB, BC, CA respectively, and form the triangle XYZ, prove that

$$\rho=2\sigma\sin(\omega-\theta),$$

where ω is the Brocard angle of the triangle ABC, σ is radius of the First Lemoine Circle of ABC, and ρ is the radius of the Cosine Circle of the triangle XYZ.

822. (K. K. RANGANATHA AIYAR, M. A.) :—If a circle cut an ellipse in A, B, C, D whose eccentric angles are $\alpha, \beta, \gamma, \delta$ then the power of any point P of the ellipse varies as

$$\sin\tfrac{1}{2}(\theta-\alpha).\sin\tfrac{1}{2}(\theta-\beta)\times\sin\tfrac{1}{2}(\theta-\gamma)\sin\tfrac{1}{2}(\theta-\delta).$$

Hence or otherwise show that the equation to an ellipse may be put in the form $S_1^{\frac{1}{3}}+S_2^{\frac{1}{3}}+S_3^{\frac{1}{3}}=0$, where ABC is a maximum inscribed $\triangle$ and S_1, S_2, S_3 the circles of curvature at A, B, C.

823. (K. K. RANGANATHA AIYAR, M. A.) :—If P_1 denote the parabola of closest contact with an ellipse at A, we may put the equation to the ellipse in the form $P_1^{\frac{1}{4}}+P_2^{\frac{1}{4}}+P_3^{\frac{1}{4}}=0$, ABC being any inscribed $\triangle$.

824. (F. H. V. GULASEKARAM, B.A.) :—With reference to the circle of curvature of the conic $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$ at the vertex A of the triangle of reference, prove the following :—

(i) ρ (the radius of curvature)

$$=\frac{Rabc}{lmn}\left[\frac{m^2+n^2-2mn\cos A}{a^2}\right]^{3/2}$$

(ii) The perpendicular distances β, γ of the centre of curvature from the sides CA, AB are given by the equations

$$\frac{\beta}{n-m\cos A}=\frac{\gamma}{m-n\cos A}=\frac{\rho}{(m^2+n^2-2mn\cos A)^{\frac{1}{2}}}$$

(iii) The equation to the radical axis of the circle of curvature and the circumcircle of the triangle of reference is

$$\frac{\beta/b+\gamma/c}{\beta/m+\gamma/n}=\frac{m^2+n^2-2mn\cos A}{al}$$

825. (ENQUIRER) :—Salmon [*Conic Sections*, Appendix on Pascal's theorem] states that the two of points denoted by

$$\left\{\begin{matrix} ab,\ de,\ cf \\ cd,\ fa,\ be \\ ef,\ bc,\ ad \end{matrix}\right\} \text{ and } \left\{\begin{matrix} ab,\ cd,\ ef \\ de,\ fa,\ bc \\ cf,\ be,\ ad \end{matrix}\right\}$$

are conjugate with respect to the conic. Prove that this is so.

826. (A. C. L. WILKINSON) :—If P_1, P_2, P_3, P_4, P_5, P_6, are six points on a conic and $\lambda=(123456)$, $\mu=(126453)$, $\nu=(123546)$, $\rho=(126543)$ are four intersecting Pascal lines corresponding to hexagons taken in the orders of the suffices as stated, prove that the cross-ratio $(\lambda\mu,\ \nu\rho)$ is equal to

$$\frac{(32,\ 16)(31,\ 26)}{(34,\ 56)(35,\ 46)}$$

where $(pq,\ rs)$ denotes the cross-ratio of the pencil subtended by the points P_p, P_q, P_r, P_s at any point of the conic.

827. (A. C. L. WILKINSON) :—If a skew surface is defined by

$$x=x,\ y=bx+\beta,\ z=cx+\gamma,$$

where b, c, β, γ are functions of t, and if the axis of x is the generator corresponding to $t=0$, the origin the central point of this generator, and $z=0$ the tangent plane at the origin; then the hyperboloid of closest contact along the generator $t=0$ is given by

$$2c'\beta'^2 z=c'\gamma''y^2+2c'^2\beta' xy-b''\beta' z^2-(c'\beta''-c''\beta')yz$$

where the values of the differential coefficients of b, c, β, γ are for $t=0$.

161908
600 008.